Problem Set 2

Student handout

Counting, events, and shared birthdays

100 points + 10 optional challenge points

Name: ______________________ **Due:** ____________

Instructions. Identify the outcomes, their probabilities, and the event being counted. State whether order matters and repetition is allowed. Give exact answers except where decimal approximations are requested. A calculator or short program may be used for Problem 7. Show enough work that the model and the calculation can be checked.

I. Probability spaces: outcomes need weights

1. A weighted finite space [12 points]

Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, let $\mathcal{F} = 2^\Omega$ be all subsets, and suppose $\mathbb{P}(\{i\}) = ci$ for $i = 1, \ldots, 6$.

(a) Find c. Explain why these masses define a probability space when $\mathbb{P}(E)$ is the sum of the masses in E. (4)

(b) Let $A = \{2, 4, 6\}$ and $B = \{4, 5, 6\}$. Find $\mathbb{P}(A)$, $\mathbb{P}(B)$, $\mathbb{P}(A \cap B)$, and $\mathbb{P}(A \cup B)$. Check the last answer by inclusion–exclusion. (4)

(c) A student says $\mathbb{P}(A) = |A|/|\Omega| = 1/2$. Explain the error and state the extra assumption that would make this counting rule valid. (4)

2. The same recorded outcomes, two experiments [10 points]

Experiment I flips two fair coins independently and records the number K of heads. Experiment II chooses an integer uniformly from $\{0, 1, 2\}$.

(a) Give the ordered sample space for Experiment I and the probability of each ordered outcome. Then give the probabilities of the recorded values $K = 0, 1, 2$. (4)

(b) Find the probability of recording at least one in each experiment. Why does the common recorded sample space $\{0, 1, 2\}$ not force the two answers to agree? (3)

(c) After Experiment II records k, choose uniformly among the two-coin strings with exactly k heads. Find the probabilities of HH, HT, TH, TT. Does this reproduce Experiment I? (3)

II. Permutations and binomial coefficients

3. What distinguishes two selections? [12 points]

Explain the counting rule in each part.

(a) Choose a three-person committee from eight students. (3)

(b) Assign president, secretary, and treasurer from those eight students, with no person holding two offices. (3)

(c) Choose a three-person committee from those students and designate one member as chair; the other two have no distinct roles. (3)

(d) Arrange the seven letters of `BALLOON`, treating repeated copies of the same letter as indistinguishable. (3)

4. Why binomial coefficients appear [12 points]

(a) Prove Pascal's identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for $1 \le k \le n-1$ by a counting argument, not by manipulating factorials. (4)

(b) Prove $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ by counting all subsets of an n-element set in two ways. (4)

(c) Find the coefficient of x^3 in $(1 + 2x)^6$. Explain both the binomial coefficient and the power of 2 in your answer. (4)

III. Inclusion–exclusion: cardinalities and probabilities

5. Overlaps among divisibility conditions [12 points]

Let A, B, C be the integers from 1 through 300 divisible by $4, 6, 10$, respectively. Here "through 300" includes both endpoints.

(a) Find the sizes of the three sets, their three pairwise intersections, and their triple intersection. (4)

(b) How many integers are divisible by at least one of $4, 6, 10$? How many are divisible by none? (4)

(c) How many are divisible by exactly two of $4, 6, 10$? Explain how you remove the triple intersection from the pairwise counts. (4)

6. Can these event probabilities occur? [14 points]

Three events have the following probabilities; no independence is assumed.

Event	A	B	C	$A \cap B$	$A \cap C$	$B \cap C$	$A \cap B \cap C$
Probability	10/20	9/20	8/20	4/20	3/20	2/20	1/20

(a) Find the probability of at least one event and the probability of none. (4)

(b) Find the probability of exactly one event. (4)

(c) Verify that the data define a possible model by giving nonnegative probability masses for the eight disjoint membership regions and checking that they sum to one. (6)

IV. Birthday collisions

7. A classroom birthday model [16 points]

Assume n labeled people have independent birthdays, each uniform over 365 days. Ignore leap days.

(a) Specify $\Omega, \mathcal{F}$, and the probability of each individual outcome. For $1 \le n \le 365$, count the outcomes with all birthdays distinct. (4)

(b) Derive an exact formula for the probability $q(n)$ of at least one shared birthday. What is the answer when $n > 365$? (4)

(c) Compute $q(22)$ and $q(23)$ to four decimal places. Explain why these values establish the smallest class size for which a collision is more likely than not. (4)

(d) For $n = 23$, compute the probability that someone else shares one designated member's birthday. Explain why it is much smaller than $q(23)$. (4)

8. Three people: inclusion–exclusion checks the product [12 points]

In a toy calendar of ten equally likely days, three labeled people have independent birthdays. Let E_{ij} be the event that people i and j share a birthday.

(a) Find $\mathbb{P}(E_{12})$, the probability of each pairwise intersection among E_{12}, E_{13}, E_{23}, and the probability of their triple intersection. Explain the event described by an intersection. (5)

(b) Use inclusion–exclusion to find the probability of at least one match. Check it by counting the complement. (4)

(c) A student computes the no-match probability as $(9/10)^3$ by multiplying the three pairwise no-match probabilities. Explain why that multiplication is not justified. (3)

Optional challenge: nobody gets their own hat [10 extra points]

The n labeled hats of n people are returned as a uniformly random permutation. Use inclusion–exclusion to show that the probability that no one gets their own hat is

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Explain the count for an intersection of k specified correct-hat events, and evaluate the probability when $n = 4$.